機械所博士班資格考　工程數學　試題

April 2006

1. (16 %) Solve the differential equation.

(1) (5 %) $y' - Ay = -By^2$ (A and B are positive constants)

(2) (5 %) $x^2y'' - 3xy' + 4y = 0$

(3) (6 %) $y'' + y = \sec x$ (by variation of parameters)

2. (17 %) Find a general solution of the nonhomogeneous linear system.

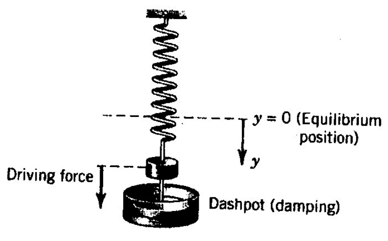
(1) (8 %) $y' = Ay + g = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$

(2) (9 %) $y' = Ay + g = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

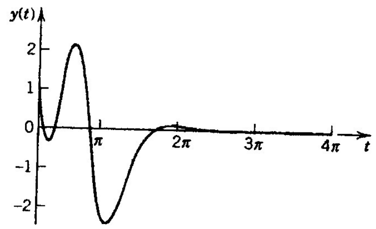
3. (17 %) Solve the initial value problem by means of **Laplace transforms**

$$y''+2y'+2y = r(t), \quad \begin{cases} r(t) = 10\sin 2t & if\ 0 < t < \pi \\ r(t) = 0 & if\ t > \pi \end{cases} ; \quad y(0) = 1,\ y'(0) = -5$$

This is a damped mass-spring system with a sinusoidal driving force acting during the interval $0 < t < \pi$ only.



Mechanical system



Output (solution)

Linear Algebra and Vector Calculus

4. Find the inverse $\mathbf{A}^{-1}$ of (17%)

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

5. Find the eigenvalues and eigenvectors of the matrix (17%)

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

6. Find the work done by the force $\mathbf{F} = 2xy^3 \sin z\,\mathbf{i} + 3x^2y^2 \sin z\,\mathbf{j} + x^2y^3 \cos z\,\mathbf{k}$ in the displacement around the curve of intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $(x-1)^2 + y^2 = 1$. (Hint: By the Stokes's Theorem) (17%)

7. (17%) Solve the following partial differential equation for $u(x, t)$ by Fourier series.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L,\ 0 < t < \infty);$$
$$u(x,0) = f(x), \quad (0 < x < L);$$
$$u_x(0,t) = 0,\ u_x(L,t) = 0, \quad (0 < t < \infty).$$

8. (17%) Solve the two-dimensional wave problem by using double Fourier series

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (0 < x < a,\ 0 < y < b);$$

that satisfies the boundary $u = 0$, on all the boundary;

and two initial conditions $u(x, y, t) = f(x, y)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y)$.

9. (17%) (a) State whether the given function is even or odd. Find its Fourier series. Sketch the function and some partial sums. (Show the details of your work)

$$f(x) = \begin{cases} k & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

(b) Use (a) to show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots = \frac{\pi}{4}$.