Engineering Mathematics

PhD Candidate Qualification Examination

Institute of Mechanical Engineering, Yuan Ze University

Apr. 2002

Part I (Ordinary Differential Equation and Laplace Transform)

1.Solve the ordinary differential equation

$y'' + 4y = x^2 \sin 2x$ (17%)

2.Using the Laplace transform, solve the following problem

$F(t) = \sin t$ if $0 < t < \pi$ and $F(t) = 0$ if $\pi < t < 2\pi$;

and $F(t+2\pi)=F(t)$, find the $L\{F(t)\}$ (17%)

3.Using the inverse Laplace transform, find the

$L^{-1}\{\frac{1}{s^3(s^2+1)}\}$ (17%)

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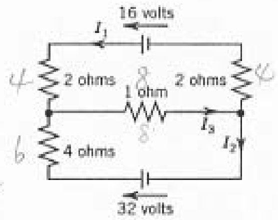
(每題 17 分)

4. Using Kirchhoff's laws, find the currents in the following networks by Cramer's rule and Gauss elimination.



5. A small body B moves on a disk toward the edge, its position vector being $\vec{r}(t) = t\vec{b}$. The disk is rotating counterclockwise with constant angular speed ω=1, and $\vec{b}$ is a unit vector rotating with the disk. Find the tangential acceleration $\vec{a}_{\tan}$ and normal acceleration $\vec{a}_{norm}$ of B.

6. Consider the motion of a compressible fluid in a region R having no sources or sinks in R. If the flow through a small rectangular box W of dimensions Δx, Δy, Δz with edges parallel to the coordinate axes. W has the volume $\Delta V = \Delta x \Delta y \Delta z$. Let $\vec{v} = [v_1, v_2, v_3] = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be the velocity vector of the motion. Assume $\vec{v}$ is continuously differentiable vector functions of x, y, z, and t. Please derive the continuity equation of a compressible fluid flow $\frac{\partial \rho}{\partial t} + div(\rho\,\vec{v}) = 0$.

7. The general expression of the Fourier transform of $f(x)$ is in the following: **(17%)**

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Please explain the following question and find the Fourier transform of the following function $f(x)$

(a) What are the differences of the Laplace transform and Fourier transform? And, why did we need to learn the Fourier transform?

(b) $f(x) = e^{-2x}$ if $x > 0$ and $f(x) = 0$ otherwise.

8. Please explain the following question and find the partial differential equation. **(17%)**

(a) What kinds of problems lead to ordinary differential equations and to partial differential equations?

(b) Find a solution $u(x, y)$ of the partial differential equation $u_{xx} - u = 0$.

9. The heat equation as shown in equation (1) which gives the temperature $u(x, y, z, t)$ in a body of homogeneous material. **(17%)**

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u \quad \text{, where } c^2 = \frac{k}{\sigma \rho} \qquad (1)$$

Here c^2 is the thermal diffusively, k the thermal conductivity, σ the specific heat, and ρ the density of the material of the body. $\nabla^2 u$ is the Laplacian of u, and with respect to Cartesian coordinates x, y, z.

As an important application, let us first consider the temperature in a long thin bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis (as shown in Fig. 1) and is perfectly insulated laterally, so that heat flows in the x-direction only.

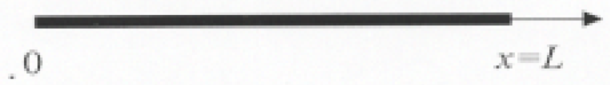


Fig. 1 Bar under consideration

Then u depends only on x and time t, and the heat equation becomes the one-dimensional heat equation as shown in equation (2).

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

We shall solve (2) for some important types of boundary and initial conditions. We begin with the case in which the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the boundary conditions in the following equation.

$$u(0, t) = 0,\ u(L, t) = 0 \text{ for all } t, \qquad (3)$$

and the initial temperature in the bar at time $t = 0$ is $f(x)$, so that we have the initial condition in the following equation.

$$u(x, 0) = f(x) \qquad (4)$$

We can solve the entire problem in the following equations.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \ (\text{where } \lambda_n = \frac{cn\pi}{L}) \qquad (5)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (\text{where } n=1, 2, \ldots) \qquad (6)$$

<u>Here is the question</u>: Please find the temperature $u(x,t)$ in a laterally insulated copper bar 100 cm long if the initial temperature is $200\sin(5\pi x/100)^\circ C$ and the ends are kept at $0^\circ C$. How long will it take for the maximum temperature in the bar to drop to $80^\circ C$? Physical data for copper: density $8.92 gm/cm^3$, specific heat $0.092 cal/(gm^\circ C)$, thermal conductivity $0.95 cal/(cm \sec^\circ C)$.